1.1.4 teaser Symmetric functions

- Recall generating functions: given a sequence $[a_n]_{n\geq 0}$ ($\mathbb{C}$-values)
the corresponding generating function is $\sum_{n\geq 0} a_n x^n$
$\mathbb{C}[[x]]$ denotes the ring of formal power series.

ex $a_n = 1$ $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$a_n = p(n)$ number of integer partitions
$\sum_{n=0}^{\infty} p(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$
variable x is tracking the size of the partition

- to keep track of more information we consider generating functions in more variables $\vec{x} = (x_1, x_2, \dots)$ coefficient

$$f(x_1, x_2, \dots) = \sum_{\alpha=(\alpha_1,\alpha_2,\dots)} c_\alpha \underbrace{x_1^{\alpha_1} x_2^{\alpha_2} \cdots}_{\text{monomial}}$$

$\mathbb{C}[[\vec{x}]]$ ring of multivariate power series

- π acts on $\mathbb{C}[[\vec{x}]]$ as follows

$$\pi f(x_1, x_2, \dots, x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}, x_{n+1}, \dots)$$

Definition 5 $f \in \mathbb{C}[[\vec{x}]]$ is symmetric if $\pi f = f$ for all $\pi \in G_n$ for all n

example 4 symmetrizing a monomial

$$x_1^2 x_2 \longrightarrow x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + \cdots$$

Definition 6 If λ is a partition $(\lambda_1, \lambda_2, \dots, \lambda_k)$, the monomial symmetric function of λ:

$$M_\lambda(\vec{x}) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k}$$

sum over all monomials having exponents $\lambda_1 \lambda_2 \cdots \lambda_k$

$M_{21}(\vec{x})$

* If f is symmetric the coefficients c_α only depend on the underlying set of exponents
so $\{M_\lambda\}$ form a basis

Definition 7 The ring of symmetric functions $\Lambda = \operatorname{span}_{\mathbb{C}}(M_\lambda)$

* Newton polynôme monic avec n racines

$(X+r_1)(X+r_2)\cdots(X+r_n)$

$= X^n + X^{n-1}(\quad) \qquad + r_1 r_2 \cdots r_n$

base élémentaire

* $\frac{1}{1-X}$

$\frac{1}{1-X}\ \frac{1}{1-Y}$

example 5 let G be graph with k vertices $V(G) = \{1, 2, \dots, k\}$

let $F_G(x_1, x_2, \dots) = \sum_{c: V(G) \to \{1,2,3,\dots\}} x_{c(1)} x_{c(2)} \cdots x_{c(k)}$

where the sum is over a proper colorings of the vertices of G with colors $1, 2, 3, \dots$

↑ if $(i,j) \in E(G)$ then $c(i) \neq c(j)$ (adjacent vertices get different colors)

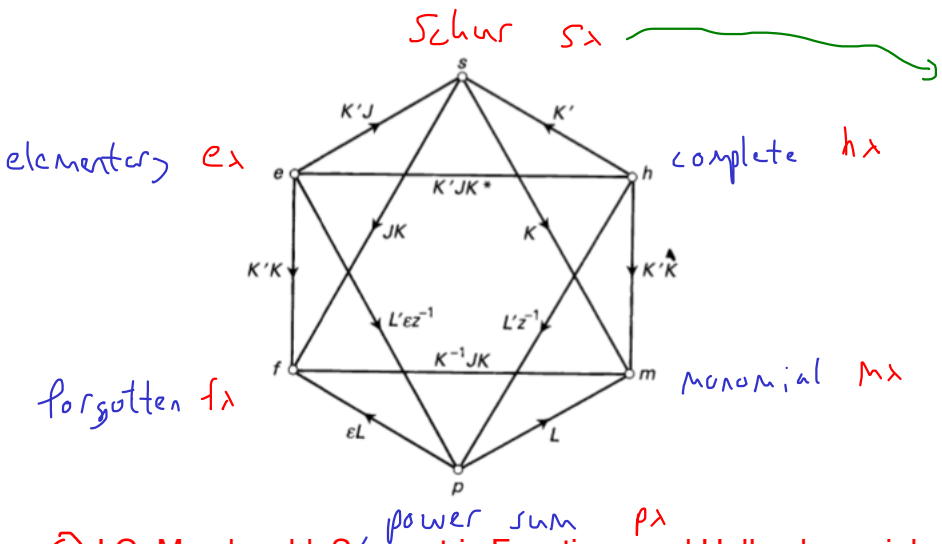
ex G  $F_G(\vec{x}) = x_1x_2 + x_1x_3 + \cdots$

Observation: $f_G(\vec{x})$ is a symmetric function

If we evaluate $F_G(\vec{x})$ at $x_1 = x_2 = \cdots = x_N = 1$ other $x_i = 0$

then $F_G(\underbrace{1,1,\dots,1}_{N},0,\dots) = \#$ proper colorings of G with N colors.

* We will study this ring Λ and its different bases:



$s_1 = x_1 + x_2 + \cdots$

$s_{11} = m_{11} = x_1x_2 + x_1x_3 + \cdots$

$s_2 = m_2 + m_{11} = x_1^2 + x_2^2 + \cdots + x_1x_2 + x_1x_3 + \cdots$

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We will study how to express a symmetric function in different basis

* $(x_1 + x_2 + \cdots)(x_1 + x_2 + \cdots) = ?$

there is a deep connection between the representation theory of G_n and symmetric functions: to each representation we can associate a symmetric function

example (continued) for the defining representation of G_2 the associated symmetric function is $(x_1 + x_2 + \cdots)(x_1 + x_2 + \cdots)$

→ we will see that the expansion of such a symmetric function in the Schur basis encodes the decomposition of the original representation into irreducibles.

$$(x_1+x_2+\cdots)(x_1+x_2+\cdots) = x_1^2 + x_2^2 + \cdots + 2x_1x_2 + 2x_1x_3 + \cdots$$
$$= m_2 + 2m_{11}$$
$$= (m_2 + m_{11}) + (m_{11})$$
$$= 1\,m_2 + 1\,m_{11}$$

1.2 Group representations

1.2.1 The symmetric group G_n (you can actually generate with two elements HW1)

$\pi \in G_n$

one line notation: $\pi_1 \pi_2 \cdots \pi_n$ $\quad \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$

cycle notation: $\cdots (i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i)) \cdots$ $\quad (\pi^k(i) = i)$

* cycle type of π is integer partition of length of cycles
$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ $\quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ $\quad \sum \lambda_i = n$

* two permutations are conjugate iff they have the same cycle type
$\pi = \cdots (a_1, a_2, \ldots, a_k) \cdots$ $\quad \sigma\pi\sigma^{-1} = \cdots (\sigma(a_1), \ldots, \sigma(a_k)) \cdots$

* Given a partition λ of n, denote conjugacy class by K_λ
$$K_\lambda = \{ \pi \in G_n \mid \text{cycle type } \pi \text{ is } \lambda \}$$

Proposition (size of conjugacy classes in G_n)
If $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$ of n then $|K_\lambda| = \frac{n!}{z_\lambda}$ where $z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots n^{m_n} m_n!$

($1^{m_1} 2^{m_2} \cdots n^{m_n}$: frequency notation)

pf HW1 or see book.

Λ^n_R essentially functions symétrique degré n sur R $(R = \mathbb{Q})$

R-module ($\mathbb{Q}$-vector vectoriel)

$af + bg \in \Lambda^n_R$

$f \in \Lambda^m_R$, $g \in \Lambda^n_R$ $\quad f \cdot g \in \Lambda^{m+n}_R$

$\Lambda_R = \Lambda^0_R \oplus \Lambda^1_R \oplus \cdots$

Λ_R est un R-algèbre gradué

Partitions

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ $\quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$

$\mathrm{Par}(n)$ partitions de n $\quad p(n) = |\mathrm{Par}(n)| \sim n\, e^{\pi\sqrt{2n}}$

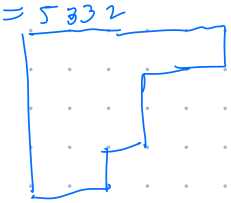
1, 11, 21, 3, 111, 31, 22, 211, 4, 1111

- $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_\ell$ taille de λ $\quad \ell(\lambda) = \ell$
- notation exponentielle $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \dots)$
- diagramme de Ferrer de λ

$\lambda = 4431$



conjuguée de λ $\quad \lambda' = 5332$



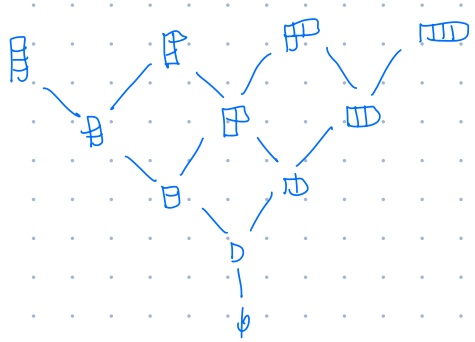
$m_i(\lambda') = \lambda_i - \lambda_{i+1}$

$\lambda'_1 = \ell(\lambda)$

$\lambda_1 = \ell(\lambda')$

Ordres dans les partitions

① $\lambda \subseteq \lambda$ si $m_i \geq \lambda_i$ pour tout i treillis de Young



la fonction de rang de Y est

$$F(Y, X) = \sum_{n \geq 0} p(n) x^n = \prod_{i \geq 1} (1 - x^i)^{-1}$$

(2) ordre de dominance $\leq$ si $|\mu| = |\lambda|$ $\mu \leq \lambda$ si

$\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ pour tout i

4 | | 3 3

3 | | |

(3) ordre lexicographique renversé

$\mu \overset{L}{\leq} \lambda$ $\mu = \lambda$ ou $\exists i$

$\mu_1 = \lambda_1, \ldots, \mu_i = \lambda_i$, $\mu_{i+1} < \lambda_{i+1}$

ex) partition

partition

- $\mathrm{Im}(x)$ i plus grande tel que $\lambda_i \geq i$

$$m_\lambda = \sum_\alpha x^\alpha$$

α permutations distinctes $\alpha = (\alpha_1, \ldots)$

$m_0 = 1$

$m_1 = x_1 + x_2 + \cdots$

$m_2 = x_1^2 + x_2^2 + \cdots$

- $\dim \Lambda^n = p(n)$

$\{m_\lambda \mid \lambda \vdash n\}$ base de Λ

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fonctions symétriques élémentaires

$$e_n = m_{1^n} = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \qquad (e_0 = 1)$$

e_1, e_2

$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$

$(A)_{ij}$ integer matrix Ferrers support

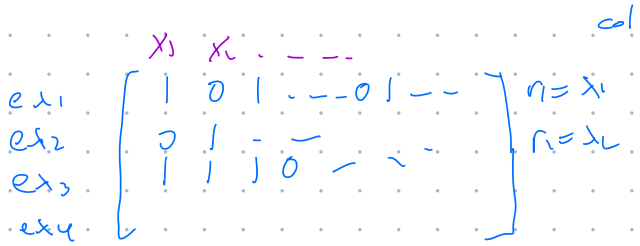
row sums $r_i = \sum_j a_{ij}$ $\quad \mathrm{row}(A) = (r_1, r_2, \dots)$

col sums $c_j = \sum_i a_{ij}$ $\quad \mathrm{col}(A) = (c_1, c_2, \dots)$

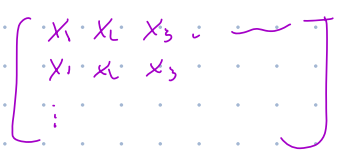
Prop $\lambda \vdash n$, $\alpha = (\alpha_1, \alpha_2, \dots)$ weak comp

$$e_\lambda = \sum_{\mu \vdash n} M_{\lambda\mu} m_\mu$$

$M_{\lambda\mu}$ = # 0/1 matrices A satisfying $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$



λ: ball labeling



Cor $M_{\lambda\mu}$ = # integer matrices symmetric

i.e. $M_{\lambda\mu} = M_{\mu\lambda}$

Proof A matrix $\in \{0,1\}$ has $\mathrm{row}(A) = \lambda$ iff $\mathrm{row}(A^t) = \mu$
$\mathrm{col}(A) = \mu$ $\quad \mathrm{col}(A^t) = \lambda$

- $f = \sum_\lambda a_\lambda m_\lambda$ if $a_\lambda \geq 0$ we say f is *n-positive*
  if $a_\lambda \in \mathbb{N}$ does it have a combinatorial interpretation?
- $f = \sum a_\lambda m_\lambda$ $a_\lambda \in \mathbb{Z}$ f is *n-integral* algebraic interpretation
- e are n-positive

Série génératrice $Z = \sum_\lambda c_\lambda u_\lambda$ — u_λ base de Λ
c_λ fonction de $\lambda \in \mathrm{Par}(n)$ — $c_\lambda \in R$, $R = \Lambda(y)$

Prop $\prod_{i,j}(1+x_i y_j) = \sum_{\lambda,\mu} M_{\lambda\mu} m_\lambda(x) m_\mu(y)$

$= \sum_\lambda m_\lambda(x) e_\lambda(y)$

Notation $[x^\alpha] f$ coeff de x^α dans f

Pr $[x^\alpha y^\beta] \prod_{i,j}(1+x_i y_j)$

$\prod_{i,j}(x_i y_j)^{a_{ij}}$

(Th fondamental fonctions symétriques)

Thm Soient $M_{\lambda\mu} = 0$ sauf si $\lambda \le \lambda'$, $M_{\lambda\lambda'} = 1$

Alors $(e_\lambda(x))_{\lambda \in \mathrm{Par}(n)}$ forme une base de Λ^n.

Cor $e_1, e_2, \dots$ sont algébriquement indépendants et engendrent Λ comme une $\mathbb{Q}$-algèbre

$\Lambda = \mathbb{Q}[e_1, e_2, \dots]$

Pr Supposons $M_{\lambda\mu} \neq 0$, il existe matrice A de $(0,1)$ t.q. $\mathrm{lig}(A) = \lambda$, $\mathrm{col}(A) = \mu$

$A = \begin{bmatrix} 1&0&0&1&0&0&1 \\ 1&0&0&1&1&1&0 \\ 0&1&0&0&1&0&0 \end{bmatrix}$ soit A' ← justifié

$\#$ lignes $A \le \#$ colonnes A'

$\lambda_1 + \lambda_2 + \dots + \lambda_i \le \lambda'_1 + \lambda'_2 + \dots + \lambda'_i$

$\Rightarrow \lambda \le \lambda'$

A' est la seule matrice $\begin{matrix}\text{ligne}(A')=\lambda \\ \text{col}(A')=\lambda'\end{matrix}$ $M_{\lambda\lambda'}=1$

- Pour tout ordre de $Par(n)$ compatible avec $\leq$

$$M = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{matrix}\lambda' \\ \lambda^2 \\ \vdots \\ \\ \end{matrix}$$

$(\lambda')' \ (\lambda^2)' \ \dots$

M est inversible (comme $\det(M)=1$ ce qui est une base de $\Lambda_{\mathbb{Z}}$)

- comme $\{e_\lambda \mid \lambda \in Par\} = \{e_1^{a_1} e_2^{a_2} \dots \mid a_i \geq 0 \ \sum a_i < \infty\}$

donc ind. croissant est $\Leftrightarrow$ indépendant algébrique de e_i

$e_1 = M_1$ $\quad e_{11} = M_2 + 2M_{11}$ $\quad e_{111} = M_3 + 3M_{21} + M_{111}$

$e_{21} = M_{21} + 3M_{111}$

$e_3 = M_{3111}$

$$E(t) = \sum_{n\geq 0} e_n t^n = 1 + e_1 t + e_2 t^2 + \dots = \prod_{i\geq 1} (1+x_i t) \in \Lambda[[t]]$$

Fonction homogène complète

$$h_k = \sum_{\lambda \vdash k} M_\lambda = \sum_{1\leq i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} x_{i_2} \dots x_{i_k}$$

$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots$

$\Rightarrow h_k(1,1,\dots,1) = h_k(1^n) = \left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ (n fois)

$j_2 = i_1+1,\ j_3 = i_2+1 \to j_k = i_{k-1}+1$ $\quad 1 \leq j_1 < j_2 < \dots < j_k \leq n+k-1$

Plus généralement $\lambda \vdash n$ $\alpha = (\alpha_1, \alpha_2, \dots)$ composition faible de n

$$h_\lambda = \sum_{M \vdash n} M_{\lambda\alpha} M_M$$

$M_{\lambda\alpha} = \#$ $\mathbb{N}$-matrices tel que $\begin{matrix}\text{ligne}(A)=\lambda \\ \text{col}(A)=\alpha\end{matrix}$

Preuve (exo) $h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_r}$

$$\begin{bmatrix} \frac{1}{1-x_1} & \frac{1}{1-x_2} & \cdots \\ \frac{1}{1-x_1} & \frac{1}{1-x_2} & \cdots \\ \vdots & & \end{bmatrix}$$

$$\begin{array}{c} \\ h_{\lambda_1} \\ h_{\lambda_2} \\ \vdots \\ h_{\lambda_l} \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & & \\ 3 & 0 & 2 & 0 & \cdots & \\ 2 & 2 & 0 & 0 & & \\ 1 & 0 & 0 & 3 & 0 & \cdots \end{array} \\ \\ c_1 = \alpha_1 \quad c_2 = \alpha_2 \cdots \end{array} \begin{array}{l} n = \lambda_1 \\ r = \lambda_l \end{array}$$

choisir $x_1^{a_{i1}} x_2^{a_{i2}} \cdots$ de chaque h_{λ_i} $\quad \sum_j a_{ij} = \lambda_i$

$\sum_i a_{ij} = \alpha_j$

Cor $M_{\lambda\mu} = M_{\mu\lambda}$

* Il y a une dualité entre e_λ et h_λ

$$(1+X)^n = \sum_{k\geq 0} \binom{n}{k} X^k$$

$$\frac{1}{(1-X)^n} = \sum_{k\geq 0} \binom{-n}{k} (-1)^k X^k = \sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) X^k$$

Prop $$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda\mu} M_{\lambda\mu} m_\lambda(x) m_\mu(y)$$

$$= \sum_{\lambda} m_\lambda(x) h_\lambda(y)$$

ex $h_1 = m_1$

$h_{11} = 2m_{11} + m_2$

$h_2 = m_{11} + m_2$

$h_{111} = 6m_{111} + 3m_{21} + m_3$

$h_{21} = 3m_{111} + 2m_{21} + m_3$

$h_3 = m_{111} + m_{21} + m_3$

$$H(t) := \sum_{i\geq 0} h_i t^i = 1 + h_1 t + h_2 t^2 + \cdots \in \Lambda[[t]]$$

$$= \prod_{i\geq 1} \frac{1}{1 - x_i t}$$

Involution

Comme $\Lambda = \mathbb{Q}[e_1, e_2, \dots]$ on peut définir un homomorphisme $f : \Lambda \to \Lambda$ avec les valeurs $f(e_i)$

Soit $\omega : \Lambda \to \Lambda$ $\quad \omega(e_i) = h_i$ pour $i \geq 1$

$\Rightarrow \omega(e_n) = h_n$

Thm ω est une involution, $\omega^2 = 1$ $\quad (\omega(h_n) = e_n)$

Preuve $\quad H(t) = \sum_{n \geq 0} h_n t^n = \prod_{i \geq 1} (1 - x_i t)$

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} (1 + x_i t)$$

on a que $H(t) E(-t) = 1$

donc on extrait $[t^n]$ des deux côtés

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0 \qquad (\star)$$

Si $\sum (-1)^i u_i h_{n-i} = 0$ pour tout $n \geq 1 \Rightarrow u_i = e_i$

On applique ω

$$\sum_{i=0}^{n} (-1)^i h_i \omega(h_{n-i}) = (-1)^n \sum_{j=0}^{n} (-1)^j \omega(h_j) h_{n-j} = 0$$

$\Rightarrow \omega(h_i) = e_i$ ☒

Cor $\{h_\lambda \mid \lambda \vdash n\}$ est une base de Λ^n. Ensuite $h_1, h_2, h_3, \dots$ sont algébriquement indépendants $\Lambda = \mathbb{Q}[h_1, h_2, \dots]$

Preuve ω est un automorphisme et $\omega(e_\lambda) = h_\lambda$, comme e_λ est une base donc h_λ est une base aussi.

Def $f_\lambda := \omega(m_\lambda)$ fonctions symétriques oubliées

exercice calculer f_{11}, f_{21} base symétrique

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fonctions de puissance symétriques

Rappel $\sum_{i\geq 1} \frac{1}{i} x^i = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots = \log(1-x)^{-1}$

$\left(\frac{d}{dt}\left(\sum_{i\geq 1} \frac{1}{i} x^i\right) = \sum_{j\geq 0} x^j = \frac{1}{1-x} \right)$

$\int \frac{1}{1-x} = \log(1-x)^{-1}$

$\exp\left(\sum_{i\geq 1} \frac{1}{i} x^i\right) = \frac{1}{1-x}$

Rappel partition d'un ensemble

Π

partition ordonnée d'un ensemble

$1=2=3$

$1=23 \quad 2=13 \quad 3=12$

$1-2-3$

Def $p_n = m_n = \sum_i x_i^n$ $\quad p_0 = 1$

$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$

Prop $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$

$$p_\lambda = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu$$

$k \leq \ell(n)$

$R_{\lambda\mu}$ nombre de partitions ordonnées $\Pi = (B_1, \ldots, B_k)$ de $[\ell]$

$$\mu_j = \sum_{i \in B_j} \lambda_i \qquad 1 \leq j \leq k \qquad B_r = \{ j \mid i_j = r \}$$

Dém $[x^\mu]\, p_\lambda = [x^\mu]\left(\sum x_i^{\lambda_1}\right)\left(\sum x_i^{\lambda_2}\right)\cdots$

Cor $R_{\lambda\mu} = 0$ sauf si $\lambda \leq \mu$ (ordre de dominance)

et $R_{\lambda\lambda} = \prod_i i^{r_i}(r_i!)$ $\quad \lambda = \langle 1^{r_1} 2^{r_2} \cdots \rangle$

Donc $\{p_\lambda \mid \lambda \vdash n\}$ forment une base de Λ^n

equiv. $p_1, p_2, \dots$ sont alg. indépendant

$\Lambda = \mathbb{Q}[p_1, p_2, \dots]$

Déf Si $R_{\lambda\mu} \neq 0$ il existe $\Pi = (B_1, \dots, B_\ell)$

Soit B_{i_1} block avec 1 $\quad \mu_{i_1} \geq \lambda_1$

$\mu_{i_2} \geq \mu_{i_1} \geq \lambda_1$

B_{i_1}, B_{i_2} blocks distincts avec 1 et 2

$\mu_1 + \mu_2 \geq \mu_{i_1} + \mu_{i_2} \geq \lambda_1 + \lambda_2$

$\vdots$

$\mu_1 + \mu_2 + \dots + \mu_r \geq \lambda_1 + \dots + \lambda_r$

* $\deg R_{\lambda\mu}$ n'est pas ± 1 donc $\{p_\lambda \mid \lambda \vdash n\}$ est une $\mathbb{Q}$-base de Λ^n, pas une $\mathbb{Z}$-base

* Soit $z_\lambda = 1^{\lambda_1} \lambda_1! \, 2^{\lambda_2} \lambda_2! \cdots$

permutations n type de cycle λ est $\frac{n!}{z_\lambda}$

$\varepsilon_\lambda = (-1)^{\lambda_2 + \lambda_4 + \cdots} = (-1)^{n - \ell(\lambda)}$

$$P(t) = \sum_{i \geq 1} p_i t^{i-1} = \sum_{i \geq 1} \frac{x_i}{1 - x_i t} = \sum_{i \geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t}$$

$$P(t) = \frac{d}{dt} \log H(t) = \frac{H'(t)}{H(t)}$$

$$n h_n = \sum_{r \geq 1} p_r h_{n-r}$$ (Newton)

$$n e_n = \sum_{r \geq 1} (-1)^{r-1} p_r e_{n-r}$$

Prop $\prod_{i,j} (1 - x_i y_j)^{-1} = \exp \sum_{n \geq 1} \frac{1}{n} p_n(x) p_n(y)$

$$= \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y)$$

$$\prod_{i,j} (1 + x_i y_j) = \exp \sum_{n \geq 1} \frac{1}{n} (-1)^{n-1} p_n(x) p_n(y)$$

$$= \sum_\lambda z_\lambda^{-1} \varepsilon_\lambda p_\lambda(x) p_\lambda(y)$$

Dem $\log \prod_{i,j} (1-x_i y_j)^{-1} = \sum_{i,j} \log (1-x_i y_j)^{-1}$

$= \sum_{i,j} \sum_{n \geq 1} \frac{1}{n} x_i^n y_j^n = \sum_{n \geq 1} \frac{1}{n} p_n(x) p_n(y)$

Prop $\omega p_\lambda = \varepsilon_\lambda p_\lambda$

Cor $h_n = \sum_\lambda z_\lambda^{-1} p_\lambda$

$e_n = \sum_\lambda \varepsilon_\lambda z_\lambda^{-1} p_\lambda$

$y = (t, 0, \dots, 0)$

$\prod (1-x_i t)^{-1} = \exp \sum_{n \geq 1} \frac{1}{n} p_n t^n$

$\mathbb{C}[t^n]$

Cor $z_\lambda =$ # nbre de permutations $\nu \in S_n$ qui commutent avec une permutation fixée w_λ de type λ.

Formule exponentielle (Ch 5)

$$F(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!} \qquad G(x) = \sum g(n) \frac{x^n}{n!}$$

① C'est quoi $F(x) G(x)$?

② C'est quoi $F(G(x))$ si c'est bien défini

③ C'est quoi $e^{F(x)}$

$f : \mathbb{N} \to K$ $\quad \mathbb{E} f(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!}$

(produit)

Prop Étant donné $f, g : \mathbb{N} \to K$, soit $h : \mathbb{N} \to K$

$$h(\#X) = \sum_{(S,T)} f(\#S) g(\#T) \qquad (*)$$

somme (S, T) partages ordonnés de X en 2 blocs

$\phi \subset S \subseteq X$, $T = X \setminus S$

donc $\mathbb{E} h(x) = \mathbb{E} f(x) \, \mathbb{E} g(x)$

Preuve $h(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) g(n-k)$

ex $\#X = (S, T)$ et ordonnés S et T

$$\sum_{n\geq 0} h(n) \frac{x^n}{n!} = \frac{1}{(1-x)^2}$$

* On peut itérer (*)

$$E_h(x) = \prod_{i=1}^{k} E_{f_i}(x)$$

Formule de composition

$f : \mathbb{P} \to K$ ($\mathbb{P} = \{1, 2, 3, 5, \dots\}$)

$g : \mathbb{N} \to K$ $g(0) = 1$ soit $h : \mathbb{N} \to K$

$$h(\#S) = \sum f(\#B_1) \cdots f(\#B_k)\, g(k) \qquad \#S > 0 \quad ☆$$

$\Pi = \{B_1, B_2, \dots, B_k\} \in \Pi(S)$

$h(0) = 1$

somme partages de S donc

$$E_h(x) = E_g\Big(\underbrace{E_f(x)}_{\sum_{n\geq 1} f(n)\frac{x^n}{n!}}\Big) \qquad (0)$$

Pr $E_g(E_F(x)) = \sum_{k\geq 0} \frac{g(k)}{k!} (E_f(x))^k$

fixe k dans ☆

$h(n) = \sum_k h_k(n)$ ($f(0) = 1$)

$\{B_1, \dots, B_k\}$ Bi non vide, il y a $k!$ façon de les ordonnées donc

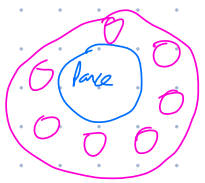
$$E_{h_k}(x) = \frac{g(k)}{k!} E_f(x)^k$$

* structures (graph orienté) sont union disjointe de composantes connexes (peut-être avec une structure additionnelle)

ex) bijection table ronde $E_f(X) = \sum_{n\geq 1} (n-1)! \frac{X^n}{n!} = \log(1-X)^{-1}$

$E_S(X) = 1 + \log(1-X)^{-1}$

$1 + \log(1 - \log(1-X))^{-1}$



* Cas plus commun $g(k) = 1$ (aucune structure dans les composantes)

Cor (formule exponentielle) Étant donné $f : \mathbb{P} \to K$ connexe

soit $h : \mathbb{N} \to K$

$$h(\#S) = \sum_{\pi = [B_1, \dots, B_k] \in \Pi(S)} f(\#B_1) \cdots f(\#B_k) \qquad \#S \geq 0$$

$h(0) = 1$

alors $E_h(X) = \exp E_f(X)$ (*)

Cor Si $f : \mathbb{P} \to K$, $h : \mathbb{N} \to K$ sont liées par (*) alors

$$h(n+1) = \sum_{k=0}^{n} \binom{n}{k} h(k) f(n+1-k)$$

$$f(n+1) = h(n+1) - \sum_{k=1}^{n} \binom{n}{k} h(k) f(n+1-k)$$

Pf Prend la dérivée de (*)

$$E_h'(X) = E_f'(X) E_h(X) \quad \text{et prend coeff } \left[\frac{X^n}{n!}\right]$$

ex graphe connexe 1 sommets $c(n)$ (ligne jaune)

" " " $g(n) = 2^{\binom{n}{2}}$ (ligne jaune)

$$\exp c(X) = \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{X^n}{n!}$$ (ligne jaune)

* Si on commence avec une permutation de S

$$h(\#S) = \sum_{\#C \in G(S)} f(\#C_1) \cdots f(\#C_k) \cdot g(k) \qquad \#S \geq 0$$

$h(0) = 1$ sous $C_1, \dots, C_k \to C_k$ ordre les décompositions de π

$$E_h(X) = E_g\left(\sum_{n\geq 1} f(n) \frac{X^n}{n}\right)$$

Pr : Il y a $(p-1)!$ façons d'ordonner cycliquement un ensemble à p éléments

$$h(\#S) = \sum_{\pi = [B_1, \dots, B_k] \in} \left(\#B_1 - 1\right)! \, f(\#B_1) \cdots \left(\#B_k - 1\right)! \, f(\#B_k) \quad \text{si } \Delta$$

Cor (△△)

$$h(\#S) = \sum_{\pi \in \mathcal{O}(r)} f(\#C_1) \cdots f(\#C_k) \qquad \#S \geq 0$$

$$h(0) = 1$$

$$\Rightarrow \quad E_{h(X)} = \exp\left(\sum f(n) \frac{X^n}{n} \right) \qquad (\star)$$

Soit $f(n) = p_n(x)\, p_n(y)$ et $h = 1$ dans $(\star)$

$$\sum_{n \geq 0} h(n) = \exp\left(\sum p_n(x) p_n(y) \frac{1}{n} \right)$$

$$h(n) = \sum_{\omega \in \mathcal{O}_n} p_{p(\omega)}(x)\, p_{p(\omega)}(y)$$

$$= \sum c_\lambda \, p_\lambda(x) \, p_\lambda(y)$$

$$c_\lambda = \#\{\omega \in S_n \mid p(\omega) = \lambda\} = \frac{n!}{z_\lambda}$$

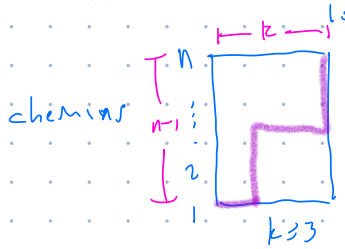
Cor

$$h_n = \sum_\lambda z_\lambda^{-1} \, p_\lambda$$

$$e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda z_\lambda^{-1} \, p_\lambda$$

$$e_k(x_1,\dots,x_n) = \sum_{1\le i_1<\dots<i_k\le n} x_{i_1}\cdots x_{i_k} \qquad \begin{cases} e_k(1,\dots,1) = \binom{n}{k} \\ h_k(1^n) = \binom{n+k-1}{k} \end{cases}$$

$$h_k(x_1,\dots,x_n) = \sum_{1\le j_1\le j_1\le\dots\le j_k\le n} x_{j_1}\cdots x_{j_k}$$



$$h_3(x_1,x_2,x_3,x_4) = \cdots + x_1x_3^2 + \cdots$$

Into q-analogues

$$[n]_q = 1+q+\cdots+q^{n-1} = \frac{1-q^n}{1-q}$$

$$[n]! = [n][n-1]\cdots[2][1]$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)} \in \mathbb{N}[q]$$

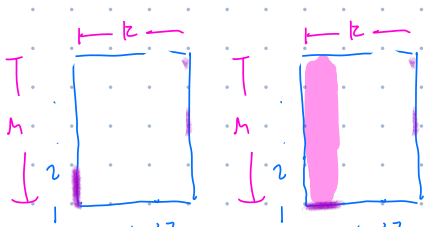
$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

par recurrence $\begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{N}[q]$

Prop $\sum_{\substack{\lambda,\ \ell(\lambda)\le n \\ \ell(\lambda)\le k}} q^{|\lambda|} = \begin{bmatrix} n+k \\ k \end{bmatrix}$

Preuve recurrence

$$\begin{bmatrix} n+k \\ k \end{bmatrix} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix} + q^n\begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}$$



Specialization

Def Soit R un algèbre sur $\mathbb{Q}$ commutatif, une specialisation est un homomorphisme $\varepsilon : \Lambda \to R$ $(\varepsilon(1) = 1)$

ex substitution des variables

Def Soit Λ_n polynômes symétriques dans $\mathbb{Q}[x_1, \dots, x_n]$

$$r_n : \Lambda \to \Lambda_n \qquad f \mapsto f(x_1, \dots, x_n) = f(x_1, \dots, x_n, 0, 0, \dots)$$

* $\{r_n(m_\lambda) \mid \lambda \in \mathrm{Par}_n\}$, $\{r_n(p_\lambda) \mid \lambda' \in \mathrm{Par}_n\}$, $\{r_n(h_\lambda) \mid \lambda' \in \mathrm{Par}_n\}$ $\{r_n(e_\lambda) \mid \lambda' \in \mathrm{Par}_n\}$ bases de Λ_n
* on peut aussi définir $w_n(e_\lambda) = h_\lambda \quad \lambda' \in \mathrm{Par}(n)$
* $ps_n(f) := f(1, q, q^2, \dots, q^{n-1})$ spécialisation principale
* $ps(f) := f(1, q, q^2, \dots) \in \mathbb{Q}(\!(q)\!)$

  spécialisation principale stable
* $ps_n^1(f) = f(\underbrace{1, 1, \dots, 1}_{n}) = f(1^n)$

base	$ps_n(b_\lambda)$	$ps(b_\lambda)$	$ps'_n(b_\lambda)$
m_λ			$\binom{n}{\ell(\lambda)}\binom{\ell(\lambda)}{m_1\ m_2\ m_3\ \dots}$
p_λ	$\prod_{i=1}^{\ell}\frac{1-q^{n\lambda_i}}{1-q}$	$\prod_i \frac{1}{1-q^{\lambda_i}}$	$n^{\ell(\lambda)}$
e_λ	$\prod_i q^{\binom{\lambda_i}{2}}\begin{bmatrix} n \\ \lambda_i \end{bmatrix}$	$\prod_i \frac{q^{\binom{\lambda_i}{2}}}{(1-q)\cdots(1-q^{\lambda_i})}$	$\prod \binom{n}{\lambda_i}$
h_λ	$\prod \begin{bmatrix} n+\lambda_i-1 \\ \lambda_i \end{bmatrix}$	$\prod \frac{1}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i})}$	$\prod \left(\binom{n}{\lambda_i}\right)$

Spécialisation : exponentielle

$ex : \Lambda \to \mathbb{Q}[t]$ $\quad ex(p_n) = t\,\delta_{1n}$

Prop a) $ex(f) = \sum_{n\geq 0} f(x_1, \dots, x_n)\frac{t^n}{n!}$

Produit scalaire

- $\langle\ ,\ \rangle : \Lambda \times \Lambda \to \mathbb{Q}$
- bases (u_λ), (v_λ) sont duales si $\langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu}$

Def $\langle m_\lambda, h_\mu\rangle = \delta_{\lambda\mu}$

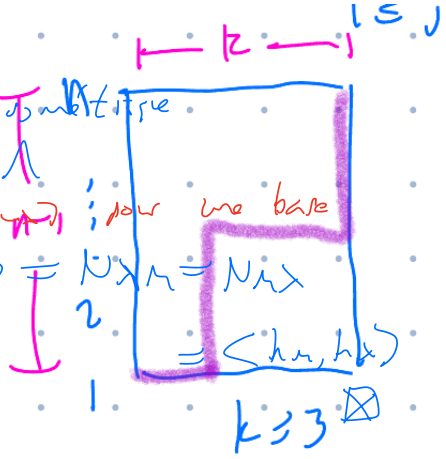
Prop Le produit scalaire $\langle\ ,\ \rangle$ est symétrique

$\langle f, g\rangle = \langle g, f\rangle$ pour $f, g \in \Lambda$

Preuve Suffit de montrer $\langle u_\lambda, v_\mu\rangle = \langle u_\mu, v_\lambda\rangle$ pour une base

$\langle h_\lambda, h_\mu\rangle = \langle \sum_\nu M_{\lambda\nu} m_\nu, h_\mu\rangle = M_{\lambda\mu} = N_{\lambda\mu}$

$= \langle h_\mu, h_\lambda\rangle$



Lemme Soit (u_λ) et (v_λ) deux bases de Λ tq $u_\lambda, v_\lambda \in \Lambda^n$ pour λ partout, alors (u_λ) et (v_λ) sont deux bases duales ssi $\sum_t u_\lambda(x) v_\lambda(y) = \prod (1 - x_i y_j)^{-1}$

Pr Algèbre linéaire $(m_\lambda) = (T)(u_\lambda)$ $\quad h_\mu = (N)(v_\nu)$

$$\delta_{\lambda\mu} = \langle m_\lambda, h_\mu \rangle = \sum_{\rho,\nu} (T)_{\lambda\rho} (N)_{\mu\nu} \langle u_\rho, v_\nu \rangle$$

$$I = T A N^t \qquad A_{\rho\nu} = \langle u_\rho, v_\nu \rangle$$

$A = I$ ssi $I = T N^t$ ssi $I = T^t N$

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda m_\lambda(x) h_\lambda(y)$$

$$= \sum_{\rho,\nu} (T^t N)_{\rho\nu} u_\rho(x) v_\nu(y)$$

$$= \sum_\rho u_\rho(x) v_\rho(y) \quad \square$$

Cor $\langle P_\lambda, P_\mu \rangle = z_\lambda \delta_{\lambda\mu}$ donc (P_λ) forment une base orthogonale de Λ (par orthogonale)

Q) Est-ce qu'il y a une base orthonormale naturelle ?